Green Scheduling for Energy-Efficient Operation of Multiple Chiller Plants

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***Abstract*— In large building systems, such as a university campus, the air-conditioning systems are commonly served by chiller plants, which contribute a large fraction of the total electricity consumption of the campuses. The power consumption of a chiller is highly affected by its Coefficient of Performance (COP), which is optimal when the chiller is operated at or near full load. For a chiller plant, its overall COP can be optimized by utilizing a Thermal Energy Storage (TES) and switching its operation between COP-optimal charging and discharging modes. However, uncoordinated mode switchings of chiller plants may cause temporally-correlated high electricity demand when multiple plants are charging their TES concurrently. In this paper, a *Green Scheduling* approach, proposed in our previous work, is used to schedule the chiller plants to reduce their peak aggregate power demand while ensuring safe operation of the TES. We present a scheduling algorithm based on backward reach set computation of the TES dynamics. The proposed algorithm is demonstrated in a numerical simulation in Matlab to be effective for reducing the peak power demand and the overall electricity cost.**

I. INTRODUCTION

Effective energy management for building facilities is becoming increasingly important in view of rising energy costs. Chilled water plants are a common means of air-conditioning for large scale building systems in which a central plant (or several plants) serve the cooling load for multiple buildings (e.g. in a university campus). Figure 1 shows an example of such a system. It shows the chilled water distribution system for the University of Pennsylvania's campus in Philadelphia, USA. The buildings are the demand side of the campus as they are responsible for its cooling load whereas the chiller plants are considered as operating on the supply side of the campus as they meet the cooling demand by supplying chilled water to the buildings. Over 4 million gallons of chilled water ($\sim 42\,^\circ$F) are pumped by several chiller plants (highlighted in Fig 1) throughout the campus to the various buildings (shown in blue) through an underground mains. The chilled water loop is supplied by two main plants (MOD 6 & MOD 7) and one emergency plant MOD 5. Each of the plants contains several chillers. Chilled water is flowed throughout campus 24 hours a day, all year round, as a result of which the power consumption of these plants contribute a large fraction of the total electricity consumption of the campus. To provide some perspective, on a hot summer day (also called a "red day"), the peak power consumption of the campus can exceed 108 megawatts (MW). Of this, MOD 7 chiller plant alone consumes about 26 MW when operating at full capacity. The electricity consumption of all the chiller plants operating at full capacity accounts for more than 30% of the total peak power.

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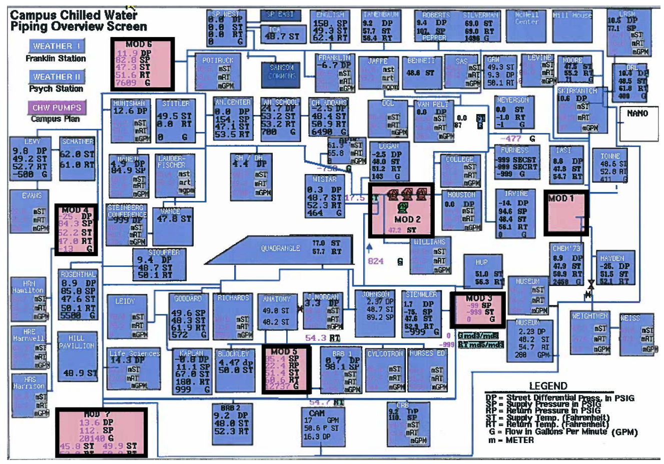

Fig. 1. Chilled water distribution system at the University of Pennsylvania. Chiller plants (highlighted and shown in pink) are labelled as MOD 1-7 and buildings are shown in blue. The power consumption of MOD 6 and MOD 7 alone accounts for more than 30% of the power consumption of the campus.

Reducing the peak power of multiple chiller plants has huge economic benefits especially under a demand-charge based electricity pricing scheme. In a peak-demand pricing policy, a large commercial electricity customer is charged not only for the amount of electricity it has consumed but also for its maximum demand over the billing cycle. The unit price of the peak demand-charge is usually very high, up to 240 times in some cases [1] and even more.

Operating multiple chiller plants can be tricky, especially since these systems vary enormously in size and configuration. Most operators use procedures inherited from the design sequence, or modified to meet some particular concern having nothing to do with peak reduction and energy conservation. The performance of a chiller plant (and of individual chillers) is given by its coefficient of performance (COP). The COP is the ratio of the total heat removed by the plant (cooling load served) to its power consumption. A higher COP is desirable since it means that the plant is more efficient. The COP of a chiller is not fixed and varies non-linearly with the part load ratio (PLR) of the chiller ([2]). The PLR of a chiller is the ratio of its cooling load on the chiller to the design load (maximum available

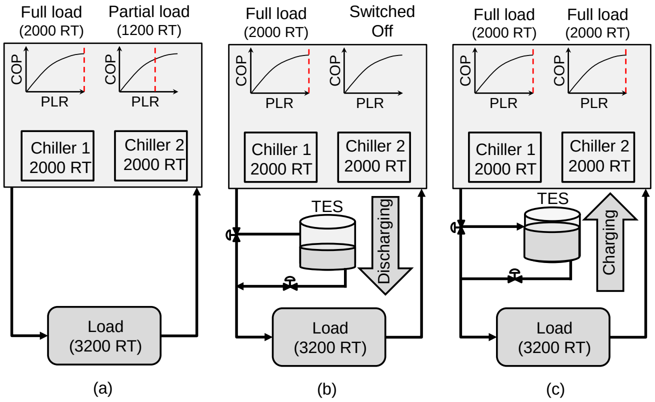

Fig. 2. Example showing benefits of adding thermal energy storage to a plant. (a) Chiller plant with no TES operates at a lower COP (b) TES in high COP discharging mode (c) TES in high COP charging mode

capacity). For example if the current load on a chiller is 1000 Refrigeration Tons (RT)[1] and the capacity (maximum load) of the chiller is 1500 RT, then the PLR of the chiller will be $1000/1500 = 0.67$.

The operation of a chiller plant can be improved by the use of a Thermal Energy Storage (TES) system. In the simplest case, a TES system for cooling is just storage of chilled water in large tanks or rock caverns. Based on the size of the TES, it can provide long term (greater than 10 hours) or short term (less than 2 hours) cooling to the load by discharging the chilled water. Short-term TES can improve the COP of the chiller plant. This can be seen in the toy example shown in Figure 2. Consider a chiller plant containing two chillers with a cooling capacity of 2000 RT each. The COP-PLR curve for each chiller is also shown next to it. We consider two cases, one without any thermal storage and the other with a TES system. Let us say, in both the cases the total load that the plant must meet is 3200 RT. In the first case (Figure 2(a)), without any storage, this can be achieved by fully loading chiller 1 to its maximum capacity of 2000 RT, therefore, the PLR of chiller 1 will be 2000/2000 = 1. In order to meet the remaining load of 1200 RT, chiller 2 must be operated at a PLR of 1200/2000 = 0.6, reducing the overall COP of the plant. Compare this with the case when there is a TES system installed (Figure 2(b) & (c)). In this case, chiller 1 can still operate at full load (of 2000 RT) but the presence of TES allows the operator to either switch off chiller 2 or operate it at a high PLR based on whether the TES is charging or discharging. When the TES is discharging, it provides the extra 1200 RT in addition to the 2000 RT being provided by chiller 1 (operating at high COP). In the charging case, both chillers are loaded to full capacity and are operated at high COP and the surplus chilled water charges the TES for future use. Therefore, with appropriate *chiller sequencing* and charging or discharging of the TES, the chiller plant can be operated at high COP.

However, when we consider multiple chiller plants, uncoordinated operation among them can cause large spikes in the total electricity consumption. In our recent papers [3], [4], we proposed *Green Scheduling* as an approach to schedule the building control systems (load side) and reduce the aggregate peak power demand while ensuring that indoor thermal comfort is always maintained. The contributions of this paper, compared to our previous work, are twofold:

1) The Green Scheduling approach is applied for the peak demand reduction of multiple chiller plants while ensuring that the cooling load is always met and the TES systems are always operated safely.
2) We present a *chiller scheduling* algorithm based on the backward reach set computation of the TES dynamics.

To the best of our knowledge, this paper is the first to consider coordinating multiple chiller plants for energy-efficient operation. The Green Scheduling approach and the proposed scheduling algorithm are shown to be effective in reducing peak power demand, hence electricity cost, by a numerical simulation.

This paper is organized as follows. Section II presents the related work in this area. We present the model of the chiller plants and the TES in Section III, and formulate the scheduling problem in Section IV. Section V reviews the schedulability results in our previous work and describes the backward reach set computation and the scheduling algorithm. Section VI presents the simulation case study in Matlab. Finally, we conclude the paper with a discussion in Section VII.

II. Background

The related work in this area can be broadly divided into two categories – chiller performance optimization and approaches for peak power reduction of HVAC systems.

The first category includes the methods which have been proposed to improve chiller operation ([5], [6], [7], [2], [8], [9]). In [5] an artificial neural network is used to obtain the chiller power consumption model which is then used for optimizing the chiller sequence. In [6], a Lagrangian method and dynamic programming are adopted to improve the COP of a single chiller plant with multiple chillers. An on-line optimal control method for multiple chiller systems in large buildings is presented in [7]. This paper also discusses the operation of the chiller under a demand based electricity pricing scheme but for a single chiller plant. A supervisory control method to improve the COP of multiple chillers in the presence of thermal energy storage is presented in [2]. [8] describes the development and evaluation of a near-optimal control rule based strategy for ice storage systems based upon simple heuristics. In [9], authors use model predictive control (MPC) to minimize the peak demand and the total energy cost of a single chiller equipped with a TES system. The MPC controller charges and discharges the TES based on the electricity cost and load consumption. In all of the above cases, the authors only consider the operation of either a single chiller or a single central chiller plant.

We will see later in Section VI that uncoordinated operation among multiple chiller plants can cause large peaks in the power consumption of the system. We focus on the problem of peak power reduction for multiple chiller plants equipped with thermal energy storage, which has not been solved before.

[1]One Refrigeration Ton is the amount of heat that must be removed to melt 1 Ton of ice in 24h. 1 RT = 3517W

The second category of related work includes the different approaches for peak power reduction of HVAC systems. MPC is a popular approach to energy efficient control for buildings [10], [11]. MPC was used to minimize building's energy consumption in [10], while peak demand reduction by MPC with real-time electricity pricing was considered in [11].

In our recent papers [3] and [4], we proposed an approach called *Green Scheduling* to reduce the peak demand of a large number of zones in multiple buildings. Green Scheduling provides a framework for scheduling control tasks with an aggregate resource envelope while ensuring their safety requirements are met. Unlike traditional real time resource scheduling which is restricted to tasks whose worst case execution times are known a priori and fixed, the execution time and other temporal parameters in Green Scheduling are time varying and dependent on the plant dynamics and environmental conditions. We have successfully used the Green Scheduling approach for peak reduction on the demand (load) side. The supply side consumes a large fraction of the system's total energy and is more responsive and efficient for peak reduction [12]. Therefore, in this paper we apply Green Scheduling to the supply side.

III. Modeling

Inside a chiller plant, multiple chillers are typically arranged in parallel, each with their dedicated pumps. The chilled water supply temperature and the load on each chiller can be adjusted. For a given cooling requirement, individual chiller loads can be controlled by using different water supply set points for constant individual flow or by adjusting individual flows for identical set points. Figure 3 shows a detailed view of a single chiller plant. The model can be broken down into the following three parts:

A. Chiller Model

As explained earlier, the COP of a chiller is defined as the ratio of the cooling load met by the chiller to the chiller's power consumption. Often, the cooling load is expressed in a dimensionless form as a part-load ratio (PLR), which is the cooling load under a given condition divided by the the design cooling capacity. The COP of a chiller is a concave function of its PLR ([13],[14]). The COP of the i^{th} chiller unit is expressed as the second order polynomial of the PLR_i

$$COP_i = a_i + b_i PLR_i + c_i PLR_i^2 \tag{1}$$

where a_i, b_i and c_i are the coefficients for the COP-PLR curve and can be determined through regression techniques once the load and power consumption are available.

The power consumption of a chiller depends on several factors. We use a generally accepted model for the power consumption of a chiller as described in [15]. The power consumption $P_{ch,i}$ of a chiller i is modelled as follows:

$$\begin{aligned}\frac{P_{ch,i}}{P_{max,i}} &= a_{0,i} + a_{1,i}(T_{cwr,i} - T_{chws,i}) \\ &\quad + a_{2,i}(T_{cwr,i} - T_{chws,i})^2 + a_{3,i}PLR_i \\ &\quad + a_{4,i}PLR_i^2 + a_{5,i}(T_{cwr,i} - T_{chws,i})PLR_i\end{aligned}$$

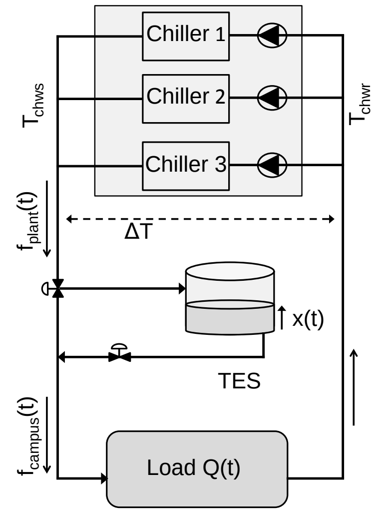

Fig. 3. Model of a single plant containing 3 chillers and a TES system. Chilled water T_{chws} flows from the chiller towards the load. Charging and discharging of the TES takes place through control valves and changes the height $x(t)$ of the water column inside the TES.

where, $T_{cwr,i}$ is the condenser water return temperature, $T_{chws,i}$ is the supply water temperature, PLR_i is the part-load ratio and $P_{max,i}$ is the maximum power consumption of the chiller. $a_{0,i} \ldots a_{5,i}$ denote regression coefficients. Given the load and the power consumption model the COP can be calculated and the coefficients of the COP-PLR curve of Eq (1) can be found.

B. Thermal Energy Storage

Thermal energy storage (TES) can be used in combination with chillers for various reasons. Long term thermal storage is typically used to shift the building loads from peak hours to off-peak hours. On the other hand, as shown by the toy example, short term thermal storage is used to increase the COP of a plant. In this paper, we assume that the thermal storage is short term. The term *charging* is often used to describe the use of storage as a buffer for cooling and *discharging* describes the use of storage to provide cooling. The control of the storage is described in the manner in which the storage is charged and discharged over time. The TES system in consideration is a chilled water storage tank. It is connected to the the chilled water distribution system through control valves which regulate the mode of the TES (Fig 3). If the chilled water flow rate being supplied from the plant, denoted by f_{plant} is greater than the required mass flow rate (f_{campus}) to meet the load of the campus, then the surplus flow charges the amount of water in the TES system. Alternatively, if the mass flow rate of water supplied from the plant is less than the required mass flow rate (i.e., $f_{plant} < f_{campus}$), then the deficit is supplied from the TES and the TES discharges. It is important to note here that charging and discharging can mean different things depending on the control of the TES. In some cases charging the TES could mean lowering the temperature of the water inside the storage. In another case, it could mean increasing the amount of water stored in the TES system. In some cases it could be a combination of both, as described in [9]. In our case (and in common practice), we consider the amount of water inside the TES system as the variable being controlled.

Therefore, we assume uniform mixing between the water supplied from the plants and the water already present in the storage. We also assume that there is no considerable heat loss between the chilled water inside the TES and the outside environment. This is a reasonable assumption since the TES is designed to serve short-term demand spikes and is charged several times in an hour. Therefore, the temperature drop across the entire volume of water inside the TES due to heat loss is small before it is charged again.

C. Load

The cooling load is the rate at which heat is removed from the conditioned space to maintain a constant (desired) space air temperature. Several factors such as the outside air temperature, heat gain due to occupants, solar heat gain, heat gain due to equipment and appliances, etc. contribute to the cooling load. At any time during the day the cooling load on the chiller plant is the sum of the cooling loads of all the buildings that it serves. Using historical data, the value of the expected total load can be predicted with high accuracy ([16],[17],[18]). Forecasts of cooling requirements and electrical use in buildings are often necessary for the control of thermal storage. In addition, forecasts can help plant operators anticipate major changes in operating modes, such as bringing additional chillers on-line. In our case we consider that the prediction for the hourly average load $Q(t)$ is available (Fig 6).

The load $Q(k)$ for the k^{th} hour can be met by either changing the chilled water supply temperature T_{chws}, by changing the flow rate f_{campus}, or both. For a given supply temperature and flow rate the value of the load determines the chilled water return temperature T_{chwr}. This is given by the following load balance equation:

$$Q(k) = f_{campus}(k) \times C_p \times (T_{chwr} - T_{chws})$$

where, C_p is the specific heat of water (in J/kgK). The temperature differential between the chilled water return and supply temperatures is commonly referred to as “Delta-T” for the chiller plant and is denoted by ΔT.

$$\Delta T = T_{chwr} - T_{chws}$$

A low value of ΔT indicates inefficient use of the chilled water being supplied from the plant. A higher chilled-water temperature differential (ΔT) in a variable flow cooling system leads to better cooling being accomplished per gallon(liter) of chilled water distributed [19]. In a variable flow system, lower level control can maintain a constant value of Delta-T while adjusting the mass flow rate of the supplied water to meet the load. In this paper, we assume that the supply water temperature T_{chws} and the temperature differential ΔT are fixed for each hour. This is a common practice by plant operators and also a simplifying assumption for our model. In a real plant, a fixed schedule can be specified for varying T_{chws} through the day.

For each chiller plant, given the predicted load profile $Q(k)$ for each hour, the chillers inside the plant can be switched on or off for the k^{th} hour. The total flow out ($f_{plant}(k)$) from the plant to the load is the sum of the flows through the individual chillers. Ensuring that the load is always met by the plant directly ensures that desired comfort is maintained inside the buildings.

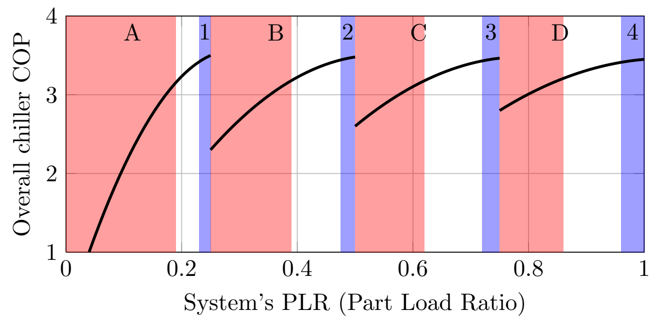

Fig. 4. COP curve for a single plant. Blue regions (1-4) are optimal COP regions and Red regions (A-D) are sub-optimal/low COP regions.

D. Assumptions

To summarize, the system model for each chiller plant makes the following assumptions:

- The thermal storage is short term.
- Average hourly predicted load is available.
- The charging and discharging of the TES occurs at a constant rate during each hour and uniform mixing of water in the TES is assumed.
- There is no heat loss between the chiller water in the TES and outside environment nor is their any heat loss through the distribution systems.
- The chilled water supply temperature T_{chws} and temperature differential ΔT are fixed.
- We also assume that when a chiller is switched on, it ramps up quite fast to meet the load. Modern chillers are capable of doing this.

IV. Problem Formulation

Having described the model for a single plant, we now formulate the problems of *chiller sequencing* within a single chiller plant (consisting of one or more chillers) and the global *chiller plant scheduling* across multiple plants for peak power minimization while ensuring the cooling load is always met.

A. Chiller sequencing with thermal energy storage

As discussed in Section I, sequencing multiple chillers to meet the cooling demand can improve the overall COP of the chiller plant. However, most of the time, at least one of the chillers is operated at a suboptimal PLR because the instantaneous cooling demand cannot be distributed optimally to any subset of the chillers. Consequently, the overall COP of the chiller plant is suboptimal. Fig. 4 illustrates the overall COP versus the PLR of a chiller plant with four equally sized chillers. The disconnected small blue regions (1-4) are operating points at or near optimal COP, while the larger red regions (A-D) correspond to operating points with low COP that we would like to avoid. For example, suppose the current cooling demand corresponds to a PLR value in the region marked B (say PLR = 0.34) in Fig. 4. If the plant only serves this demand then it will operate at a suboptimal COP. However, the chiller plant can operate at an optimal

COP (regions 1 and 2) if a TES is available in the system and is charged or discharged as required.

Increasing the number of chillers and varying their sizes will help increasing the number of high COP regions [20, p. 5.8] at the expense of increased initial investment and maintenance cost. It is suggested in [21, p. 8.3] that using short term TES, where cooling demand is smoothed out over a short time period, can improve the efficiency of the chiller plant by not exactly matching demand. This allows the chillers to be either operated at full load or switched off rather than to run continuously at part load, thus improving the overall COP.

In other words, the TES acts as an energy buffer to allow the chiller plant to switch between COP-optimal operating modes.

B. Scheduling multiple chiller plants with TES

A large campus or a cooling district often operates several chiller plants. Suppose that each chiller plant utilizes a short-term thermal storage system to improve its COP. If the plant operations are uncoordinated, it is possible that multiple of them, even all of them, operate in the charging mode concurrently, causing high electricity demand of the chiller plants. This issue is made worse by the brief power surge during the start-up of the chillers. Although the total energy consumption remains the same, the high peak demands caused by uncoordinated operation might result in a much higher electricity cost under a demand-charge pricing policy. *Green Scheduling*, proposed in our previous works [3], [4], is an approach to coordinate the operation of dynamical systems, such as chiller plants with TES, under an aggregate peak demand envelope while ensuring certain safety constraints, such as the state-of-charge of the TES. By constraining the peak demand, the demand charge, and thus the total energy cost, can be reduced. In this paper, Green Scheduling is applied to coordinate multiple chiller plants operating with chiller sequencing and limited capacity TES.

Suppose there are $m > 1$ chiller plants that we want to coordinate with Green Scheduling. At the beginning of the day, based on load prediction, each plant $i = 1, \dots, m$ will perform its own optimization to determine the two operating modes to switch between during each hour h, where $h = 0, 1, \dots, H$ and $H > 0$ is the final hour of the day:

- *Charging mode*: when the TES of the chiller plant is charged with rate $a_{i,h} > 0$;
- *Discharging mode*: when the TES of the chiller plant is discharged with rate $b_{i,h} < 0$.

At each time instant $t \geq 0$, let $x_i(t) \in \mathbb{R}$ be the charging state (height of the water column) of the TES of chiller plant i. For safe operation of the TES, x_i must be maintained between a lower threshold l_i and an upper threshold h_i at all time, i.e., $l_i \leq x_i(t) \leq h_i \; \forall t \geq 0$. Let $u_i \in \{0, 1\}$ be a binary variable which specifies the operating mode of chiller plant i. At time t, plant i is in charging mode if $u_i(t) = 1$ and is in discharging mode otherwise. Based on $x_i(t)$ and the current charging and discharging rates of all chiller plants, a scheduler will determine the operating mode $u_i(t)$ of each plant so as to reduce the peak demand by constraining the number of plants in charging modes, while ensuring that a safe charging level is maintained for each TES at any time. This control structure is illustrated in Fig. 5.

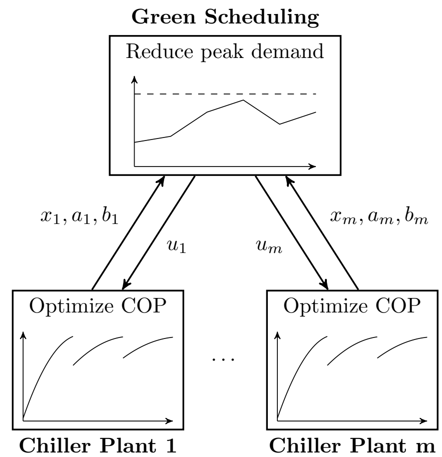

Fig. 5. Overall structure of Green Scheduling for chiller plants: each chiller plant selects its COP-optimal charging and discharging modes while the scheduler determines which mode each plant is in. The modes are selected such that the peak power demand is reduced.

The objective of Green Scheduling is to operate the chiller plants safely (in terms of the charging level of the TES) and optimally (in terms of the COP), and to smoothen and reduce the peak of the aggregate demand curve of the entire campus.

V. GREEN SCHEDULING FOR CHILLER PLANTS

During hour h of the day, i.e., $h \leq t < h + 1$, the TES of chiller plant i has a simple constant-rate dynamics:

$$\dot{x}_i(t) = \begin{cases} a_{i,h} & \text{if } u_i(t) = 1 \\ b_{i,h} & \text{if } u_i(t) = 0 \end{cases} \tag{2}$$

Green Scheduling aims to schedule the operating modes $u_i(t)$, $t \geq 0$, of all plants so that:

1) $l_i \leq x_i(t) \leq h_i$ for all t and all i; and
2) $\sum_{i=1}^{m} u_i(t) \leq k(t)$ for all t, where $k(t) \in \{0, 1, \dots, m\}$ is a time-varying peak constraint.

The peak constraint $k(t)$ specifies the maximum number of plants in charging mode at any time t.

In this section, we first review the schedulability result for Green Scheduling in our previous work ([3], [4]) to determine $k(t)$. We then present a scheduling algorithm for chiller plants based on backward reach set computation.

A. Schedulability

Let us fix the peak constraint $k(t) = k$ and the dynamics (2) of the TES of each chiller plant: $a_{i,h} = a_i > 0$ and $b_{i,h} = b_i < 0$. The system of all chiller plants is said to be *schedulable* with peak constraint k, $0 \leq k \leq m$, if there exists a schedule for the plants so that the above two conditions are satisfied. The following theorem, adapted from [3] and [4], states a necessary and sufficient condition for the system being schedulable with a given peak constraint k.

Theorem 1 ([3], [4]): For each i, define $\eta_i = \frac{-b_i}{a_i - b_i} \in [0, 1]$. The system is schedulable with peak constraint k if and only if $\sum_{i=1}^{m} \eta_i \leq k$.

It follows that the peak constraint k needs to be chosen so that $k \geq \lceil \sum_{i=1}^{m} \eta_i \rceil$ where $\lceil c \rceil$ denotes the smallest integer not less than c.

Because the dynamics (2) varies between hours of the day, we can choose a time-varying peak constraint $k(t)$ as a piece-wise constant function satisfying:

$$k(t) = k_h = \left\lceil \sum_{i=1}^{m} \frac{-b_{i,h}}{a_{i,h} - b_{i,h}} \right\rceil, \quad \forall t \in [h, h+1)$$

Although we can simply choose a constant peak constraint $k_{\max} = \max_{0 \leq h \leq T} k_h$, a tighter time-varying peak constraint will result in a smoother demand curve and potentially a lower overall peak demand.

B. Discrete-time Green Scheduling

In practice, chillers and chiller plants do not switch modes frequently due to their slow dynamics and the operation constraints of equipment. Therefore, in this paper, we consider discrete-time scheduling algorithms with a time step $\Delta > 0$, that is the operating modes of the chiller plants can only be changed at discrete time instants $\tau\Delta$ for $\tau = 0, 1, 2, \ldots$. We assume that one hour is a multiple of Δ, for example $\Delta = 15\,\text{min} = 0.25\,\text{h}$. Let T be the last time step corresponding to the last hour H of the day. Since we have discretized the time, we will write $x_i(\tau)$ and $u_i(\tau)$, $\tau = 0, 1, 2, \ldots$, to denote the charging state and the operating mode of plant i at discrete time instant $\tau\Delta$. Similar, because $k(t)$ is constant for $t \in [\tau\Delta, (\tau+1)\Delta)$, we will write $k(\tau)$ for the peak constraint during time step τ.

Define the vector of states $x = [x_1, \ldots, x_m]^T \in \mathbb{R}^m$, the binary vector of modes $u = [u_1, \ldots, u_m]^T \in \{0,1\}^m$ and the set of safe states $S = [l_1, h_1] \times \cdots \times [l_m, h_m] \subset \mathbb{R}^m$. The safety requirement reads $x(\tau) \in S$ for all τ. Let $X_f \subseteq S$ be a given set of final states, that is $x(T) \in X_f$. X_f is used to specify desired levels of the TES of the chiller plants at the end of the day. For any initial state $x(0) = x_0 \in S$, let $\mathbb{U}(k(\cdot), S, x_0, X_f)$ denote the set of all discrete-time schedules $u : \{0, 1, \ldots, T-1\} \to \{0,1\}^m$ such that:

1) $\sum_{i=1}^{m} u_i(\tau) \leq k(\tau)$ for all τ;
2) $x(\tau) \in S$ for all τ;
3) $x(T) \in X_f$.

We first characterize the set $\mathbb{U}(k(\cdot), S, x_0, X_f)$, then we will develop scheduling algorithms (belonging to $\mathbb{U}(k(\cdot), S, x_0, X_f)$) for the chiller plants.

For a time step τ, because the operating modes $u(\tau)$ of the plants are fixed until the next time step $(\tau+1)$, the charging level of each TES i at time step $(\tau+1)$ is given by the time-varying discrete-time dynamics

$$\begin{aligned} x_i(\tau+1) &= f_i(x_i(\tau), u_i(\tau), \tau) \\ &= x_i(\tau) + \begin{cases} a_{i,h(\tau)}\Delta & \text{if } u_i(\tau) = 1 \\ b_{i,h(\tau)}\Delta & \text{if } u_i(\tau) = 0 \end{cases} \end{aligned}$$

where $h(\tau)$ denotes the hour of the day corresponding to time step τ. The dynamics of all the TES are written succintly as

$$\begin{aligned} x(\tau+1) &= f(x(\tau), u(\tau), \tau) \\ &= \begin{bmatrix} f_1(x_1(\tau), u_1(\tau), \tau) \\ \vdots \\ f_m(x_m(\tau), u_m(\tau), \tau) \end{bmatrix} \end{aligned} \tag{3}$$

Given any $X \subseteq S$, we define the *backward reach set operator* (or *inverse image operator*) at time step τ

$$R_\tau^{-1}(X) := \left\{ x \in S : f(x, u, \tau) \in X \wedge \sum_{i=1}^{m} u_i \leq k(\tau) \right\} \tag{4}$$

Using this operator, we can compute a sequence of sets $\{X_\tau : \tau = 0, 1, \ldots, T\}$ as

$$X_T = X_f \tag{5a}$$

$$X_\tau = R_\tau^{-1}(X_{\tau+1}), \quad 0 \leq \tau < T \tag{5b}$$

Each set X_τ in the sequence is the set of safe states from which and from time step τ, the system state can reach the final set X_f safely under the time-varying peak constraint $k(\cdot)$. Therefore, they characterize the set $\mathbb{U}(k(\cdot), S, x_0, X_f)$ as stated in the following theorem.

Theorem 2: Suppose $x(0) = x_0 \in X_0$. A schedule $u : \{0, 1, \ldots, T-1\} \to \{0,1\}^m$ belongs to $\mathbb{U}(k(\cdot), S, x_0, X_f)$ if and only if at every time step $\tau \in \{0, 1, \ldots, T-1\}$, $u(\tau) \in \mathcal{U}_\tau(x(\tau))$, where $\mathcal{U}_\tau(x(\tau))$ is the set of valid operating modes at state $x(\tau)$ at time step τ and is defined as

$$\mathcal{U}_\tau(x) := \Big\{ u \in \{0,1\}^m : f(x, u, \tau) \in X_{\tau+1} \wedge \sum_{i=1}^{m} u_i \leq k(\tau) \Big\}, \quad \text{for } x \in X_\tau. \tag{6}$$

Proof: See the Appendix. ■

C. Green Scheduling algorithm for chiller plants

Theorem 2 dictates that at every time step τ, a safe Green Scheduling algorithm must choose $u(\tau)$ from the set $\mathcal{U}_\tau(x(\tau))$ of valid operating modes. Different algorithms differ only by how they make this decision. Generally, dynamic programming (DP) [22] can be used to compute schedule u to minimize some cost function, e.g., the total energy consumption of the chiller plants.

However, the discrete nature of mode switching of the plants make the DP problem a combinatorial optimization (Integer Programming), thus computationally expensive to solve. To alleviate this problem, a receding-horizon approach (or Model Predictive Control – MPC) [22] can be used to suboptimally solve the optimization with a rolling short horizon. In this paper, we use instead a simpler scheduling algorithm.

At each time step $\tau > 0$, only $u \in \mathbb{U}_\tau(x(\tau))$ are considered (Theorem 2). For each of them a cost is computed, which consists of two components:

- From the operation perspective, a chiller plant should not switch its operating mode (charging or discharging) too frequently. Let $D > 0$ be a given *suggested* dwell time, that is each plant should stay in a mode for at least D time steps. However, this requirement is not strict and a plant may switch its mode earlier if this is necessary to maintain a safe charging level of its TES. Thus, we define a *switching cost* $J_{sw}(u)$ as follows:

$$J_{sw}(u) = \sum_{i=1}^{m} |u_i - u_i(\tau-1)| \max(0, D - d_i(\tau-1))$$

Algorithm 1 Green Scheduling Algorithm for Chiller Plants

$d_i(0) \leftarrow 1$ for each i
Initialize $u(0)$
for $\tau = 1, \ldots, T-1$ **do**
  for all $u \in \mathbb{U}_\tau(x(\tau))$ **do**
    $J(u) \leftarrow \alpha J_{sw}(u) + \beta J_e(u)$
  end for
  $u(\tau) \leftarrow u$ with smallest $J(u)$
  for $i = 1, \ldots, m$ **do**

$$d_i(\tau) \leftarrow \begin{cases} d_i(\tau-1)+1 & \text{if } u_i(\tau) = u_i(\tau-1) \\ 1 & \text{if } u_i(\tau) \neq u_i(\tau-1) \end{cases}$$

  end for
end for

where $d_i(\tau-1)$ is the number of time steps that plant i has been staying in its current mode. If $d_i(\tau-1) \geq D$, the switching cost of plant i is 0; otherwise, the sooner the switching, the higher the incurred cost.

- We also want to reduce the aggregate energy consumption, thus we define an *energy cost* $J_e(u)$ associated with the charging modes (which consume more energy):

$$J_e(u) = \sum_{i=1}^{m} u_i P_i(\tau)$$

where $P_i(\tau) > 0$ is the power consumed by plant i in its charging mode.

The total cost $J(u)$ is a weighted sum of these two costs:

$$\begin{aligned} J(u) &= \alpha J_{sw}(u) + \beta J_e(u) \\ &= \sum_{i=1}^{m} (\alpha \, |u_i - u_i(\tau-1)| \max(0, D - d_i(\tau-1)) \\ &\qquad + \beta u_i P_i(\tau)) \end{aligned}$$

in which $\alpha \geq 0$ and $\beta \geq 0$ are appropriate weights. Often $\alpha \gg \beta$ to prevent fast mode switching.

We initialize $u(0)$ based on the value of $x(0)$: up to $k(0)$ chiller plants are in charging mode ($u_i(0) = 1$) if their initial charging levels ($x_i(0)$) are closer to the lower thresholds (l_i) than the upper thresholds (h_i). At each time step $\tau > 0$, $J(u)$ is computed for each $u \in \mathbb{U}_\tau(x(\tau))$ and which has the minimum cost will be selected to be $u(\tau)$. This scheduling algorithm is listed in Alg. 1.

The dwell time requirement is not strict since the operation of the TES to remain in its safe region is considered more critical than the switching rate of chillers. As each plant consists of multiple chillers, there can be several combinations of chiller operation inside a plant to achieve a particular mode. To prevent the same set of chillers being switched ON every time the plant switches its mode, the chillers inside a plant can be sequenced such that the switching rate of each chiller is slowed down. One such way to sequence the chillers inside a plant is to use a round-robin scheme, however chiller sequencing methods are not the focus of this work.

D. Remark

Forward and backward reachability for safety analysis and control of dynamical systems have been well studied in the literature [23], [24]. For wide classes of system dynamics, these reach sets can be computed or approximated efficiently both in discrete time and continuous time, using polytopes ellipsoids zonotopes support functions and other techniques We remark that even though we consider simplified constant-rate dynamics for TES in this paper, the reach set computation and the scheduling algorithm can be applied to any dynamics for which the backward reach set operator can be computed, for example affine dynamics [4].

TABLE I
CHILLER PLANT CONFIGURATION IN THE SIMULATION

Plant 1	3 chillers rated at 1250 RT, 1200 hp
Plant 2	3 chillers rated at 1250 RT, 1200 hp
Plant 3	3 chillers rated at 1000 RT, 900 hp
T_{chws}	5.5 °C (42°F)
T_{cwr}	20 °C (68°F)
ΔT	10 °C

VI. SIMULATION

In this section, we report the results of the simulation case study where the Green Scheduling (Section V) approach is applied for peak power reduction of multiple chiller plants with TES. We compare the performance of Green Scheduling to a TES system without Green Scheduling and to plants without a TES system.

A. Simulation Setup

The simulation was set up for three chiller plants, each containing 3 chillers. The sizing of the chillers and the configuration of the plants are summarized in Table I. The maximum power of each chiller corresponds to the case when the chiller is fully loaded. The chillers inside each plant were identically sized so that their operation is independent of the sequence in which they switch on and off. The size of one of the plants (Plant 3) was slightly smaller than the other two. Each chiller plant is equipped with its own TES system. For maximum usage, the TES was sized such that it can provide the cooling capacity of a single chiller for 1 hour. The total volume of the TES for each plant was $420\,\mathrm{m}^3$. With a cross-section area of $28\,\mathrm{m}^2$, the height of the TES system was $15\,\mathrm{m}$. As explained in Section V, safety features that minimize the risk of prematurely depleting the storage capacity are important. The lower and upper thresholds for the volume of the column of water were 20% & 90% of the total volume respectively. This meant that the height (amount) of water in the TES should always lie between $3 - 13.5$m. The temperature differential ΔT between the return and supply water temperature was fixed at 10 °C. The dwell time D and the time step $\tau\Delta$ (Section V-B) for Green Scheduling is kept as 15 mins. This ensures that the chillers do not switch rapidly between modes. Figure 6 shows the load average hourly cooling load profile for each chiller plant as explained in Section III.

B. COP Improvement

As mentioned in Sections I and IV, the presence of a short term TES system for a chiller plant can improve its COP. This was also verified through simulation. The operation of a single chiller plant (identical to Plant 1) containing 3 chillers was simulated under two different cases. The first, when no

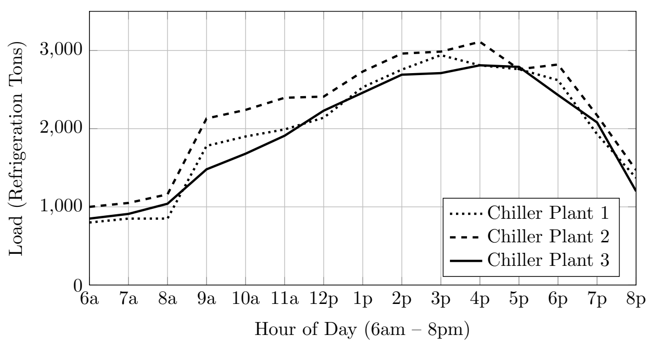

Fig. 6. Average hourly load profile for each chiller plant.

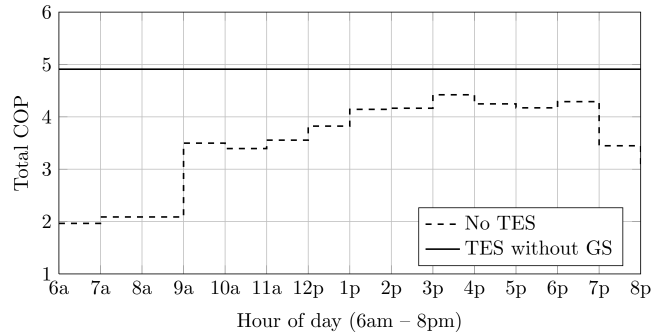

Fig. 7. Comparison of COP of a chiller plant with and without TES. Optimal COP operation is achieved when TES is present.

TES was present and the other with TES system present. The load profile used for this simulation was the load profile for Plant 1(Fig. 6). The hourly COP for the plant for both the cases has been plotted in Figure 7. It can be seen that a higher COP (4.9) is achieved in the case when the TES is present. This value corresponds to the modes when the chillers are fully loaded and is calculated as follows:

$$COP = \frac{1250RT}{1200hp} = \frac{1250 \times 3517(W)}{1200 \times 746(W)}$$

The plant with TES operates constantly at high COP since the presence of TES allows the plant to always switch between regions of high COP. In the case without TES, the COP is reduced to due non-optimal chiller sequencing in which chillers operate at sub-optimal PLR (and hence lower COP) as they always meet the cooling load.

C. Peak Power Improvement

Peak power simulation was carried out for all the three chiller plants. We ran the simulation for the entire system from 6am in the morning till 8pm in the evening. This is because, during early morning (before 6am) and late night (after 8pm) the chiller plants are usually switched off and the buildings use outside air for ventilation. Other load shifting strategies like pre-cooling the building during the night are also carried out during these hours. We do not directly take into account the effect of such measures. But since any load shifting strategy ultimately affect the hourly cooling load profile so its effect can also be included indirectly in the simulation. Our focus is on reducing the peak power during periods of high load in the day and therefore we consider the load profile from 6am-8pm.

The power consumption profile for the entire system with three chiller plants is shown in Figure 8 for three different cases. The first case considers three chiller plants but without any TES system installed. In the second case, a TES system is present for each plant but the operation between the plants is uncoordinated and there is no Green Scheduling. Each plant charges its TES to the maximum threshold, waits for it to discharge to the lower threshold and then charges it back again. The third case is the case in which Green Scheduling is implemented to coordinate the operation of the plants for peak power reduction and ensure that the TES operates safely. The results of the figure are summarized in Table II. Some interesting observations can be made from the figure.

For the first case, with no TES present, the peak power consumption is $6.4\,\mathrm{MW}$, which is lowest of all the cases. However, the total power consumption for the day is $80.9\,\mathrm{MW\,h}$, the highest among all the cases (Table II). This is expected since the system without TES simply meets the load for each hour and therefore the case when all three chillers are running at fully capacity simultaneously, thereby causing peaks in the system, does not occur. We have seen that in a system without TES, the COP is not optimal. The effect of the low COP is reflected here since the system's total power consumption being the highest for the day. The peak power consumption of the second case of TES without Green Scheduling is $7.38\,\mathrm{MW}$ and the total power consumption for the day is $66.1\,\mathrm{MW\,h}$. Even though the peak power is higher than the first case, without any TES, the total power consumption is much lower (18.3%) because of the improved COP. This clearly shows that using TES for improving the system's COP can reduce the overall power consumption but at the cost of inducing peaks in the power consumption profile. This is because of the uncoordinated operation of different chiller plants where each plant operates only to improve its own COP, independently of the other plants. This highlights the need for the Green Scheduling approach to coordinate the power consumption of multiple chiller plants.

In the third and final case, the chiller plants are coordinated using the Green Scheduling approach described in Section V. We observe that the peak power consumption is $6.48\,\mathrm{MW}$ which is 12% lower than the peak power for the TES without Green Scheduling case and almost equal to the case with no TES. The total energy consumption for TES with Green Scheduling is $61.4\,\mathrm{MW\,h}$, which is the lowest among all the cases. This can also be seen Fig. 10 which

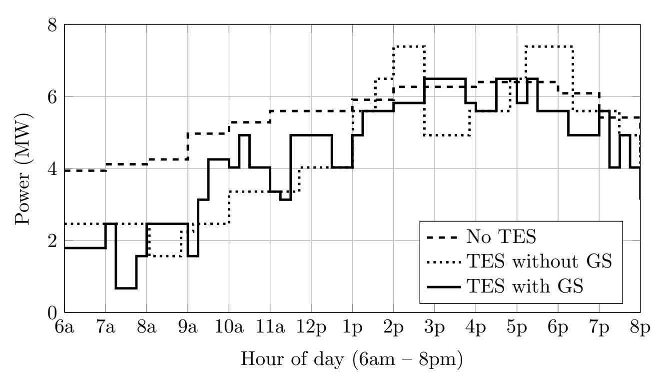

Fig. 8. Power consumption of the system for a single day for three cases. Shows how COP-optimal operation using TES can induce peaks in the power demand and Green Scheduling can lower the peak

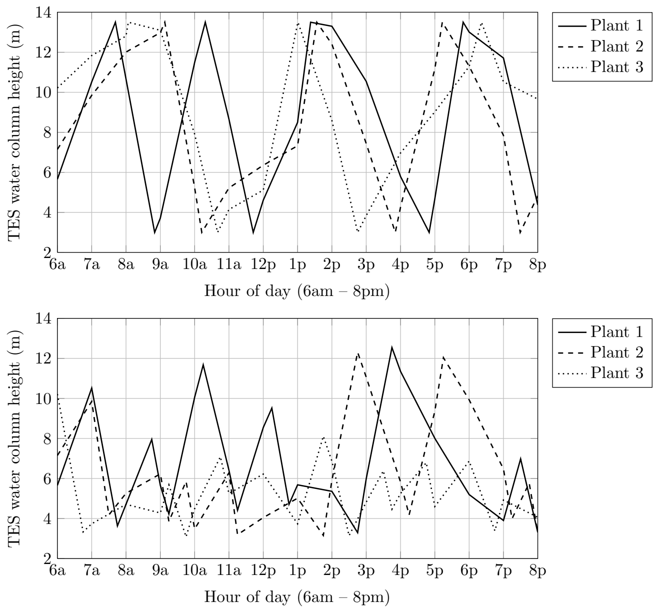

Fig. 9. Hourly TES water column heights without GS (top) and with GS (bottom). The height of the column is always maintained in the safe region.

shows the cumulative power consumption (MW h) in the three cases discussed above. The system with no TES has the highest power consumption due to a poor COP. The power consumption for the TES with Green Scheduling and TES without Green Scheduling are quite close due to high COP but the peak demand for TES with Green Scheduling is lower. Therefore, clearly Green Scheduling reduces the peak power of the system while operating with a high COP. It should be noted that this improvement is achieved by better scheduling of control systems and does not require any significant equipment additions. The benefits of this approach are echoed in the fact that under a demand based pricing scheme where Green Scheduling leads to maximum savings.

D. Demand Based Pricing

Under the widely adopted peak demand pricing policy, an electricity customer is charged not only for the amount of electricity it has consumed but also for its maximum demand over the billing cycle. The unit price of the peak demand charge is usually very high. This is to discourage

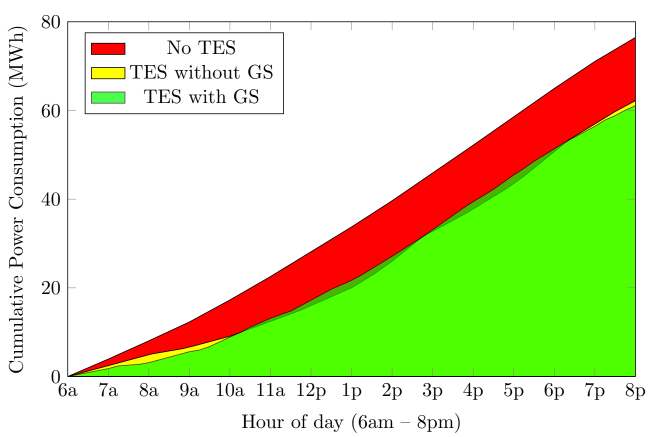

Fig. 10. Cumulative power plot showing that the total power consumption of Green Scheduling is the lowest followed closely by the TES without GS. The total consumption for no TES is quite high due to poor COP.

TABLE II
SUMMARY OF THE SIMULATION CASE STUDY

	Peak (MW)	Energy Consumption (MWh) single day	Exp. Monthly Bill ($)	% savings
No TES	6.40	**80.9**	**292,801**	-
On-Off with TES	**7.38**	66.1	274,266	6.33
GS with TES	6.48	61.4	243,461	**16.85**

TABLE III
PECO'S DEMAND PRICING RATE STRUCTURE

Block	kWh's in Block	Charges (cents per kWh)
First Block	80 × peak	24.94
Second Block	80 × peak	12.67
Third Block	Remaining	8.64

the use of electricity under peak load conditions since they can cause issues such as low quality of service and service disruptions, which affect the reliability of the grid. Table III shows an example of such a pricing scheme ([1] used by the Philadelphia based utility company PECO (Philadelphia Electrical Company). The total power consumption at the end of the billing cycle is divided into different blocks and each block is priced differently. The utility company follows declining block rate structure (Table III) with the rate for the first block being the highest and so on. What makes high peaks bad for billing is the fact that the size of the block is determined by the maximum peak in the billing cycle (a factor of 80 times the peak power). Therefore, the larger the peak demand, the larger is the number of kWhs which fall in the (most expensive) first block.

In Table II the expected bill for each of the simulation cases is computed using this pricing policy. The billing cycle is 1 month. The peak demand for the month is equal to the peak demand for the simulated day while the total monthly consumption is estimated based on the energy consumption for a single day. Green Scheduling leads to the lowest electricity bill. That's because the peak demand for Green Scheduling is very close to the lowest peak demand and the total power consumption for Green Scheduling is the lowest among all the cases. The results indicate that Green Scheduling has the potential to reduce the bill by 16.85% as compared to the case without TES and a reduction of 11.23% as compared to uncoordinated operation of TES without Green Scheduling.

The height of the water column in the TES for each plant is shown in Figure 9. It can be seen that the height of the column is always within the safe operational limits of $3-13.5$m. Lastly, the backward reach set computation described in Section V-B is very fast and takes about 5.5 s on a 2.26 GHz, dual-core machine running MATLAB R2009b.

VII. Conclusion

In this paper, we solve the problem of peak power reduction of multiple chiller plants, each equipped with a TES system. We show how uncoordinated chiller sequencing for COP improvement of multiple plants with TES can be detrimental to their operation as it can induce spikes in the cumulative power demand. High peak demands caused by such an operation can result in higher electricity costs under a demand-charge based pricing policy. The Green Scheduling algorithm coordinates the operation of the chiller plants in order to reduce their peak power demand, while ensuring that the cooling load for each plant is always met, the TES always operates safely and the system operates at a higher COP. The algorithm is based on the backward reach set computation of the TES dynamics. Through simulations, this approach is shown to be effective in reducing peak demand for a system with multiple chiller plants. Using a real demand based pricing scheme, it is shown that the Green Scheduling approach has the potential to reduce the total monthly electricity bill by almost 17% compared to a system without TES and about 10% compared to a system with TES but operating in an uncoordinated manner. To give an example, the annual electricity bill for the University of Pennsylvania campus averages about \$28 million, which clearly shows that even small improvements can lead to a large amount of savings at these scales. As the proposed Green Scheduling is largely algorithmic, its implementation is simple and compatible with existing systems.

The proposed approach presents early insight into potentially large campus-wide cost savings through the scheduling of control systems and without the need for significant equipment investment. To the best of our knowledge, this paper is the first to consider coordinating multiple chiller plants for energy-efficient operation. We aim to follow up on this effort in three directions: (a) improved chiller, TES and chilled water distribution models, (b) introduction of uncertainties in load prediction and (c) application of the proposed scheme to historical and on-line chiller data for the University of Pennsylvania campus. We are also currently investigating scheduling based on dynamic pricing models that can lead to peak-power reduction in an on-line fashion.

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Appendix

Proof of Theorem 2

Proof: Suppose that at every time step $\tau \in \{0, 1, \ldots, T-1\}$, $u(\tau) \in \mathcal{U}_\tau(x(\tau))$.

- From the definition of $\mathcal{U}_\tau(x(\tau))$ (Eq. (6)), $\sum_{i=1}^m u_i(\tau) \leq k(\tau)$ for every τ.
- We have $x(0) = x_0 \in X_0$. If $x(\tau) \in X_\tau$ for any $\tau \in \{0, 1, \ldots, T-1\}$ then by Eq. (6), $x(\tau+1) \in X_{\tau+1}$. By induction, $x(\tau) \in X_\tau \subseteq S$ for all $\tau = 0, 1, \ldots, T$, in particular $x(T) \in X_T = X_f \subseteq S$.

Therefore, by definition of $\mathbb{U}(k(\cdot), S, x_0, X_f)$, u belongs to $\mathbb{U}(k(\cdot), S, x_0, X_f)$.

Now suppose that $u \in \mathbb{U}(k(\cdot), S, x_0, X_f)$. We will prove that $u(\tau) \in \mathcal{U}_\tau(x(\tau))$ at every time step $\tau \in \{0, 1, \ldots, T-1\}$ by contradiction. Assume this were not true. If u violates the peak constraint at some time step τ, i.e., $\sum_{i=1}^m u_i(\tau) > k(\tau)$, then clearly $u \notin \mathbb{U}(k(\cdot), S, x_0, X_f)$. Thus we only need to consider u that satisfies the peak constraint $k(\cdot)$. Let τ' be the first time step at which $u(\tau') \notin \mathcal{U}_{\tau'}(x(\tau'))$. It follows from Eq. (6) that $x(\tau'+1) \notin X_{\tau'+1}$. We will show that $x(\tau) \notin X_\tau$ for all $\tau \geq \tau'+1$. If $x(\tau-1) \notin X_{\tau-1}$ for some $\tau \leq T$ then $x(\tau) \notin X_\tau$ because otherwise, by Eq. (5b) and (4), $x(\tau-1)$ must belong to $X_{\tau-1}$. Hence, by induction, $x(\tau) \notin X_\tau$ for all $\tau \geq \tau'+1$. In particular, $x(T) \notin X_T = X_f$. Therefore $u \notin \mathbb{U}(k(\cdot), S, x_0, X_f)$ which contradicts the hypothesis. ∎